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The wave equation: From eikonal to anti-eikonal approximation☆

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ABSTRACT

When the refractive index changes very slowly compared to the wave-length we may use the eikonal approximation to the wave equation. In the opposite case, when the refractive index highly variates over the distance of one wave-length, we have what can be termed as the anti-eikonal limit. This situation is addressed in this work. The anti-eikonal limit seems to be a relevant tool in the modelling and design of new optical media. Besides, it describes a basic universal behaviour, independent of the actual values of the refractive index and, thus, of the media, for the components of a wave with wave-length much greater than the characteristic scale of the refractive index.

1. Introduction

This paper is devoted to the problem of waves propagation in heterogeneous media in a special case, when the characteristic sizes of heterogeneities are much smaller than the operating wavelengths. This problem attracts now a growing attention due to the progress in the electromagnetics of metamaterials, containing the sub wavelength inclusions, e.g., the air-filled pores in plastics [1], nanoresonators [2], left-handed materials [3], metallic nanospheres in the dielectric matrix or, vice versa, dielectric nanospheres in metallic matrix [4–6]; similar problems arise in the optics of electromagnetic shock waves or localized optical missiles [7–10]. The exact analytical solutions in these problems are obtained for some special distributions of refractive indexes in heterogeneous dielectrics [11] and, thus, the development of approximative methods becomes a challenging problem. Contrarily to the well known eikonal approach in wave physics, applicable in the cases of slow variations of parameters of wave field or media along the wavelength [12], those other cases we mention can be expected to be at the opposite limit of that of the geometrical approximation, at the limit that we term as the anti-eikonal approximation.

To compare and contrast the eikonal and anti-eikonal approximations let us recall some salient features of the former approach. For simplicity, we consider a one-dimensional wave equation.

2. General framework

As it is well known, the Wave Equation can be derived from Maxwell equations for the scalar potential [13]. We consider here the one-dimensional case, $x \in \Omega \subseteq \mathbb{R}$,

$$\frac{\partial^2\phi}{\partial x^2} - \frac{n^2}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0. \tag{1}$$

Assuming that the refractive index, n, is a constant the solutions can be expressed as a suitable superposition of planes waves of the form

$$\phi(x,t) = \phi_0 e^{i(\kappa x - \omega t)},\ \kappa = \frac{n\omega}{c}, \tag{2}$$

with ϕ_0 a constant. We may introduce the wave length λ instead of the wave number and some new constants λ_0 and κ_0, accordingly:

$$\kappa = \frac{2\pi}{\lambda},\ \kappa_0 = \frac{\omega}{c} = \frac{2\pi}{\lambda_0}. \tag{3}$$

With these, we may express the plane waves as

$$\phi(x,t) = \phi_0 e^{i\kappa_0(nx - ct)}. \tag{4}$$

In the case when n is no longer a constant but varies with x, we may still look for solutions to (1) formally similar to plane waves (2) with coefficients depending on x:

$$\phi(x,t) = \phi_0(x) e^{i\kappa_0(L(x) - ct)}, \tag{5}$$

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where ϕ_0 depends now on x. This is a valid assumption for waves far away from sources [13]. The function $L(x)$ is called the optical path or the spatial-depending wave phase or, also, the *eikonal function*. It is interesting to note that the Hamiltonian structure that exists for the case n constant is preserved and we may define a Hamiltonian density functional

$$h = \frac{1}{2}\left|\frac{\partial\phi}{\partial x}\right|^2 + \frac{1}{2}\frac{n^2}{c^2}\left|\frac{\partial\phi}{\partial t}\right|^2 \tag{6}$$

that satisfies:

$$\frac{dh}{dt} = \frac{\partial}{\partial x}\Re\left(\frac{\partial\overline{\phi}}{\partial x}\frac{\partial\phi}{\partial t}\right). \tag{7}$$

Integrating over the region Ω, we define the Hamiltonian as

$$H = \int_{\Omega} h dx. \tag{8}$$

Under suitable boundary conditions on the borders of the region Ω considered, the variation in time of H, given by

$$\frac{dH}{dt} = \Re\left[\frac{\partial\overline{\phi}}{\partial x}\frac{\partial\phi}{\partial t}\right]_{\partial\Omega}, \tag{9}$$

vanishes and we have a conservation law. As a difference with the case with n constant, the time conservation of a linear momentum is not ensured. On the other hand, conservation of the positive definite functional H ensures that the variations of $\phi(x, t)$ remain bounded.

For further simplifications in the expressions, we may express the function $\phi_0(x)$ in an exponential form with exponential amplitude $A(x)$ such that

$$\phi(x, t) = e^{A(x)}e^{i\kappa_0(L(x)-ct)}. \tag{10}$$

Substituting (10) into (1) and collecting terms, we obtain

$$\left[\frac{d^2A}{dx^2} + \left(\frac{dA}{dx}\right)^2 - \kappa_0^2\left(\frac{dL}{dx}\right)^2 + \kappa_0^2 n^2 + 2i\kappa_0\frac{dA}{dx}\frac{dL}{dx} + i\kappa_0\frac{d^2L}{dx^2}\right]\phi = 0. \tag{11}$$

Assuming that ϕ is not null and splitting into the real and imaginary parts we obtain the two equations:

$$\frac{d^2A}{dx^2} + \left(\frac{dA}{dx}\right)^2 + \frac{4\pi^2}{\lambda_0^2}\left[n^2 - \left(\frac{dL}{dx}\right)^2\right] = 0, \tag{12}$$

$$\frac{d^2L}{dx^2} + 2\frac{dA}{dx}\frac{dL}{dx} = 0. \tag{13}$$

The so called eikonal approximation corresponds to the case where $n(x)$ varies slowly with space: whenever λ_0 is small, compared to the length of significant change for $n(x)$, and assuming that the amplitude A has a slow variation, the relevant term in (12) is the third one. The equation corresponding to this approximation is the eikonal equation:

$$\left(\frac{dL}{dx}\right)^2 = n^2(x), \tag{14}$$

and is the basic equation of the linear optics theory.

3. Anti-eikonal limit

Before considering the opposite limit, let us establish some additional details of the general framework. It is interesting, for instance, to substitute (10) into (8) and determine the value of H and the conditions that ensure its time-conservation. We have

$$H = \frac{1}{2}\int_{\Omega}\left[\kappa_0^2 n^2 + \left(\frac{dA}{dx}\right)^2 + \kappa_0^2\left(\frac{dL}{dx}\right)^2\right]e^{2A}dx, \tag{15}$$

and the boundary conditions that ensure conservation are those such that

$$\left[-c\kappa_0^2 e^{2A(x)}\frac{dL}{dx}\right]_{\partial\Omega} = 0. \tag{16}$$

More relevantly, Eq. (13) can be solved, at least partially, in terms of $A(x)$, to give:

$$\frac{dL}{dx} = \alpha e^{-2A(x)},\ \alpha \text{ a constant.} \tag{17}$$

We may, then substitute this into (12) to obtain:

$$\frac{d^2A}{dx^2} + \left(\frac{dA}{dx}\right)^2 + \frac{4\pi^2}{\lambda_0^2}[n^2 - \alpha^2 e^{-4A(x)}] = 0, \tag{18}$$

and we achieve, in practice, uncoupling the equations. The value of the constant α remains to be determined, using some global consideration or some boundary conditions, for instance.

To better illustrate what is meant by the anti-eikonal limit, let us call D the length of significant change for $n(x)$. We seek to establish a relation between D and λ_0. For this, we consider a new spatial variable proportional to D: $s=Dx$. Naming $\widehat{A}(s) = A(x)$, $\widehat{L}(s) = L(x)$ and $\hat{n}(s) = n(x)$, Eq. (18) becomes in the new spatial variable:

$$\frac{d^2\widehat{A}}{ds^2} + \left(\frac{d\widehat{A}}{ds}\right)^2 + \frac{4\pi^2D^2}{\lambda_0^2}[\hat{n}^2 - \alpha^2 e^{-4\widehat{A}(s)}] = 0. \tag{19}$$

The eikonal limit corresponds to $\lambda_0 \ll 2\pi D$. Let us consider now the opposite limit, and call anti-eikonal limit the case where

$$\lambda_0 \gg 2\pi D. \tag{20}$$

In that case, assuming n (or, conversely, $\hat{n}$) bounded, the relevant terms in (19) are the first two and the equation, together with (17), becomes:

$$\begin{cases} \frac{d^2\widehat{A}}{ds^2} + (\frac{d\widehat{A}}{ds})^2 = 0, \\ \frac{d\widehat{L}}{ds} = \alpha D e^{-2\widehat{A}(s)}, \end{cases} \tag{21}$$

or, reverting the change of variables,

$$\begin{cases} \frac{d^2A}{dx^2} + (\frac{dA}{dx})^2 = 0, \\ \frac{dL}{dx} = \alpha e^{-2A(x)}. \end{cases} \tag{22}$$

This set of two simultaneous equations has a remarkable property: it is independent of $n(x)$ and, thus, of the medium. In this sense it is a "universal" feature in the space – time region region to be analysed deeply. This description, however, only applies to those wavelengths of the order of λ_0 and higher: components with shorter wavelengths could not be approximated adequately in this way.

The set can be solved but it is necessary to check the validity of the assumption, namely, checking whether the third term in (19) may become relevant for large values of s and could no longer be disregarded. A way to establish this is to perform a standard multiscale analysis [14,15]. This, together with some numerical experiments, will be published elsewhere.

4. Final remarks and future work

1. First of all, let us stress that this behaviour is not restricted to the one-dimensional wave equation, although we have presented the simpler case for sake of clarity: if we consider the wave equation in three space dimensions for a scalar function,

$$\Delta\phi - \frac{n^2}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0. \tag{23}$$

with the ansatz

$$\phi(\vec{x}) = e^{A(\vec{x})}e^{i\kappa_0(L(\vec{x})-ct)}, \tag{24}$$

we obtain

$$\Delta A + (\vec{\nabla} A)^2 + \frac{4\pi^2}{\lambda_0^2}\left[n^2 - (\vec{\nabla} L)^2\right] = 0, \tag{25}$$

$$\Delta L + 2\vec{\nabla} A \cdot \vec{\nabla} L = 0. \tag{26}$$

where, as before, $\lambda_0 = 2\pi/\kappa_0$. In the anti-eikonal limit, assuming that n has a rapid variation spatial scale, homogeneously in every direction, much smaller than λ_0, the behaviour of the waves is given by the universal set of simultaneous equations:

$$\begin{cases} \Delta A + \left(\vec{\nabla} A\right)^2 = 0, \\ \Delta L + 2\vec{\nabla} A \cdot \vec{\nabla} L = 0. \end{cases} \tag{27}$$

The derivation of these equations starting from Maxwell equations, as is done in [13] for the eikonal approximation, will be presented elsewhere.

2. A similar universal description may arise in different contexts, whenever the opposite limit to the eikonal limit or WBK approximation is considered, for instance, in acoustics or in the quantum formulation of the time-independent Schrödinger equation [16].
3. What characterizes the eikonal and the anti-eikonal limits is a large disparity of the wave-length and of the characteristic length-scale of the variations of the refractive index $n(\vec{x})$. One may consider special situations where both are simultaneously impossible to consider, as is the case of waves interacting with a self-similar fractal layer, since in such a medium the spatial scales range from very small to very large [15].

5. Conclusions

We have presented the opposite situation to the eikonal limit, the anti-eikonal limit, and obtained a universal description valid as an approximation for large wave-lengths.

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